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SIMILARITY IN THE PROBLEM OF CONTACT BETWEEN ELASTIC BODIES*

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Contact between and collision of two elastic bodies, with the distance between them determined by an arbitrary, smooth, positive and positively homogeneous function, is considered. Such functions are numerous (quadratic forms and fourth degree forms are particularly special examples of such functions). Although explicity solutions of such general problem are not obtained in the end, nevertheless certain qualitative conclusions are reached (in particular, qualitative results of the Hertz theory are obtained without utilizing the explicit formulas for the potential of an oblate ellipsoid).

1. Impressing a perfectly rigid stamp into an elastic half-space. We shall deal only with the stamps, the surface of which $x_3 = f(x_1, x_2)$ is determined by a smooth, positive, positively homogeneous function of degree β, i.e.

$$f(x_1, x_2) > 0,\ \forall\ (x_1, x_2) \in R^2 \setminus \{0\} \qquad (1.1)$$
$$f(x_1, x_2) \in C^1(R^2);\ f(\lambda x_1, \lambda x_2) = \lambda^\beta f(x_1, x_2),\ \forall \lambda \geqslant 0$$

We note that the function satisfying (1.1) is fully defined by its value on the unit circle and by its degree β. When $\beta > 1$, it has a horizontal tangent plane at the zero, i.e. there is no singularity at the zero in the contact problem. We know /1/ that all stresses and displacements in the elastic half-space $x_3 \leqslant 0$, at the boundary plane of which the tangential stresses σ_{i3} $(i = 1, 2)$ are zero, can be expressed by a single harmonic function F. In particular, when $x_3 = 0$ we have

$$u_3 = 2(1-\nu) F(x_1, x_2, 0),\ \sigma_{33} = 2\mu \partial F(x_1, x_2, 0)/\partial x_3 \qquad (1.2)$$

where ν, μ denote the Poisson's ratio and shear modulus of the half-space.

Let a perfectly rigid stamp be impressed without friction into an elastic half-space. When the form of the stamp $f(x_1, x_2)$ and the impressing force P are both given, we must find the region G at the boundary of the half-space at the points at which the stamp is in contact with the half-space. Here the constant α represents the elastic approach of the bodies and the harmonic function F appears in (1.2). The quantities sought must satisfy the following conditions (∂G is the boundary of the open region G):

$$c_1 F(x_1, x_2, 0) = f(x_1, x_2) - \alpha,\ (x_1, x_2) \in G \cup \partial G \qquad (1.3)$$
$$\partial F(x_1, x_2, 0)/\partial x_3 = 0,\ (x_1, x_2) \in R^2 \setminus G$$

$F = 0$ at ∞ when $x_3 = 0$, and

$$P = c_2 \iint \frac{\partial F(x_1, x_2, 0)}{\partial x_3} dx_1 dx_2,\quad c_1 = 2(1-\nu),\quad c_2 = 2\mu$$

We note that the equation $\partial F/\partial x_3 = 0$ in (1.3) is also satisfied on ∂G. Without this condition the system (1.3) becomes indeterminate and the region G of contact cannot be determined uniquely /2/.

Theorem 1. Let the stamp surface $f(x_1, x_2)$ be defined by a positive, smooth, positively homogeneous function of degree $\beta > 1$. Let the harmonic function F_1, region G_1 and constant α_1 yield a solution to the contact problem (1.3) for the fixed force P_1. Then a solution of the contact problem for any force P will be given by the triad F, G, α, defined by the following relations:

$$F(x_1, x_2, x_3) = \rho^{-\beta} F_1(\rho x_1, \rho x_2, \rho x_3),\ (x_1, x_2) \in G \qquad (1.4)$$

if and only if

$$(\rho x_1, \rho x_2) \in G_1,\ \alpha = \rho^{-\beta}\alpha_1,\ \rho = (P_1/P)^{1/(\beta+1)}$$

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Proof. We define, for any $\lambda > 0$, the region G_λ, function $F_\lambda(x_1, x_2, x_3)$ and constant α_λ as follows:

$$F_\lambda(x_1, x_2, x_3) = \lambda^{-\beta} F_1(\lambda x_1, \lambda x_2, \lambda x_3), \ (x_1, x_2) \in G_\lambda \tag{1.5}$$

if and only if

$$(\lambda x_1, \lambda x_2) \in G_1, \ \alpha_\lambda = \lambda^{-\beta}\alpha_1$$

We shall show that F_λ is a solution of the contact problem (1.3) for a stamp of the form $f(x_1, x_2)$ in the region G_λ, with the elastic approach constant α_λ and some compressive force P_λ, which will be determined later. The definition of function F_λ implies that it is harmonic and vanishes at infinity. Taking into account (1.5), (1.3) and (1.1), we obtain

$$c_1 F_\lambda(x_1, x_2, 0) = c_1\lambda^{-\beta} F_1(\lambda x_1, \lambda x_2, 0) = \lambda^{-\beta} f(\lambda x_1, \lambda x_2) - \lambda^{-\beta}\alpha_1 =$$
$$f(x_1, x_2) - \alpha_1\lambda^{-\beta}, \ (x_1, x_2) \in G_\lambda \cup \partial G_\lambda$$

i.e. the first condition of (1.3) is fulfilled. The validity of the second condition of (1.3) is proved in the same manner. Let us find the compressive force P_λ corresponding to this solution

$$P_\lambda = 2\mu \iint_{G_\lambda} \frac{\partial F_\lambda}{\partial x_3} dx_1 dx_2 = 2\mu\lambda^{1-\beta}\lambda^{-2} \iint_{G_\lambda} \frac{\partial F_1}{\partial(\lambda x_3)} d(\lambda x_1) d(\lambda x_2) = \lambda^{-1-\beta} P_1$$

If we take

$$\lambda = (P_1/P)^{1/(\beta+1)} \tag{1.6}$$

then $P_\lambda = P$, and F_λ, G_λ and α_λ are solutions of the contact problem for this force and are found using the formulas (1.4).

Theorem 1 leads to the following qualitative corollaries (which hold under the assumptions made above):

1) The contact area a varies proportionally to the load raised to power $1/(\beta + 1)$;

2) Approach of the stamp to the half-space is proportional to the load raised to power $\beta/(\beta + 1)$.

Indeed, the definition of the region G_λ implies that the size of the contact area varies proportionally to λ^{-1} and this yields, after substituting λ from (1.6), the first assertion. The second assertion follows from the third formula of the system (1.4).

2. Contact of two elastic bodies initially touching each other.

Let us place the origin of Cartesian x_1, x_2, x_3-coordinates at the point of initial contact between two bodies. We combine the x_1Ox_2 plane with the general plane tangent to the surfaces of the bodies at the point of contact, and assume that the resultants of compressive forces lie on the x_3 axis. We shall denote the quantities referring to the body $x_3 \geqslant 0$ by the plus, and those referring to the second body by the minus sign, and make an assumption normal in such cases /1,3/, that the region G of contact is small and both bodies can therefore be replaced by half-spaces. Then the problem reduces to that of obtaining the harmonic functions F^+, F^- of the region G and constant α, satisfying the following conditions:

$$2(1-\nu^+)F^+ + 2(1-\nu^-)F^- = f(x_1, x_2) - \alpha, \ (x_1, x_2) \in G \cup \partial G \tag{2.1}$$

$F^+ = 0$ at ∞, $F^- = 0$ at ∞ when $x_3 = 0$

$$\frac{\partial F^+(x_1, x_2, 0)}{\partial x_3} = \frac{\partial F^-(x_1, x_2, 0)}{\partial x_3} = 0, \quad (x_1, x_2) \in R^2 \setminus G$$
$$\mu^+ \frac{\partial F^+}{\partial x_3} = \mu^- \frac{\partial F^-}{\partial x_3}, \quad (x_1, x_2) \in G$$
$$P = 2\mu^+ \iint_G \frac{\partial F^+}{\partial x_3} dx_1 dx_2; \quad f(x_1, x_2) = f^+(x_1, x_2) + f^-(x_1, x_2)$$

where $f(x_1, x_2)$ is a function of the distance separating the surfaces of the bodies and P is the compressive force. The last two formulas of (2.1) follow from the condition that the bodies are in equilibrium.

Lemma. Let the harmonic function F, region G and constant α all satisfy the system (1.3) for the given force and function $f(x_1, x_2) = f^+(x_1, x_2) + f^-(x_1, x_2)$ with the constants

$$c_1 = \{2[(1-\nu^+)\mu^- + (1-\nu^-)\mu^+]\}, \ c_2 = 2\mu^+\mu^-$$

and let the functions F^+ and F^- be defined by the formulas

$$F^+ = \mu^- F, \ F^- = \mu^+ F \tag{2.2}$$

Then the functions F^+, F^-, region G and constant α all satisfy the system (2.1). The lemma is proved by substituting (2.2), c_1 and c_2 into (1.3).

The lemma and Theorem 1 together yield an assertion, analogous to Theorem 1, for the contact of two elastic bodies, and this leads to the corresponding assertions 1 and 2. In particular, when $\beta = 2$, we obtain the Hertz result /1,3/ and $\beta = 4$ yields the Shtaerman result /3/.

3. Problem of collision of two bodies. Using the assumptions of the Hertz's theory of impact /4/ we obtain

$$\frac{m^+ m^-}{m^+ + m^-} \frac{d^2}{dt^2} \alpha = -P \tag{3.1}$$

where m^+ and m^- denote the masses of the colliding bodies and P is the pressure between the bodies.

Theorem 2. Let the distance between the surfaces of two colliding elastic bodies at the instant of contact be determined by a positive, smooth, positively homogeneous function of degree $\beta > 1$. Then the maximum approach α_* of the bodies and time of collision T will be proportional to the velocity v of approach of the bodies before the collision, of degree $2\beta/(2\beta + 1)$ and $-1/(2\beta + 1)$ respectively. The proof is obtained by integrating (3.1) with assertion 2 of Theorem 1 taken into account, in exactly the same manner as in the derivation of the Hertz theory /4,5/. In particular, we obtain

$$T = \sqrt{\pi} \frac{2\beta}{2\beta+1} \frac{\alpha_*}{v} \Gamma\left(\frac{\beta}{2\beta+1}\right) \Big/ \Gamma\left(\frac{4\beta+1}{4\beta+2}\right)$$

Setting $\beta = 2$ yields the result due to Hertz /4/.

We note that in case of collision of two solids of revolution the surfaces of which satisfy the condition of close contact of order $2n$ (n is a natural number), formulas analogous to those obtained above follow from the Shtaerman solution /5/. The case of axisymmetric contact problem was also studied in /6/.

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